Chords in a circle

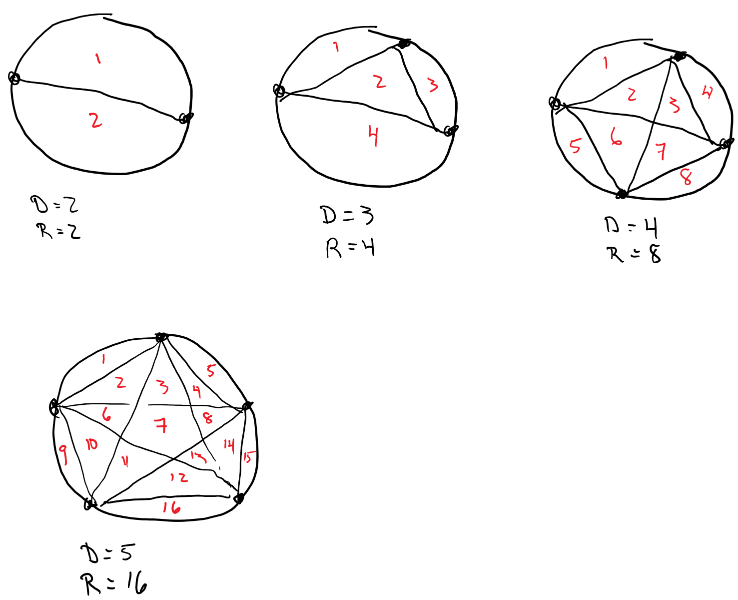
Craig: So the question about "How many pieces can n chords of a circle cut it into?" I didn't get very far.

Jenifer: Me too, either, neither. We're cutting up a disk by all the chords connecting n points on its boundary? That one.

Craig: Yes. And we're not interested in special cases where the chords are set up so lots go through a particular point like the center of the circle.

Jenifer: I agree we decided that we'd look at the case where only two intersect at any point.

Craig: So I checked the first few cases with 2, 3, 4, points and so on. Here (*Draws and counts or shares a drawing and the counts*)

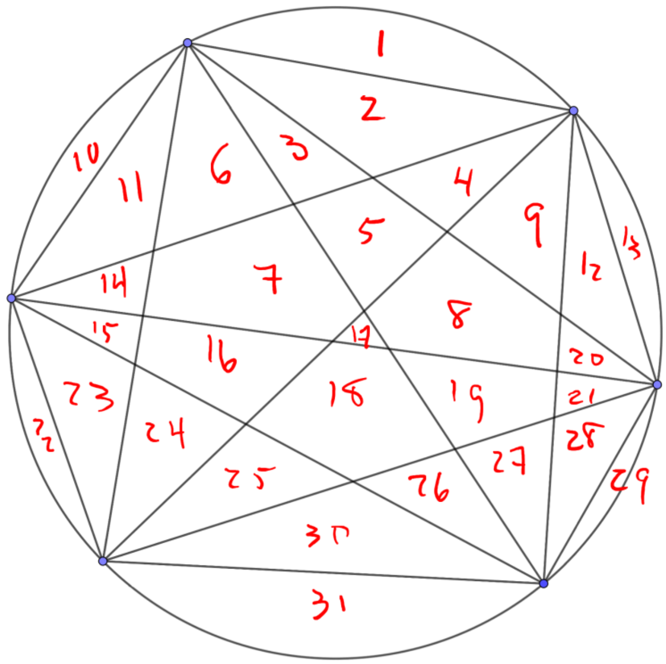


Jenifer: So it looks like 2^n.

Craig: That is $R = 2^{D-1}$? Where D is the number of dots we start with on the circle before drawing all possible chords to connect them.

Jenifer: Yes, like that. I don't see anything in the pictures that would explain an exponential. Can we look at one more with 6 dots on the circle?

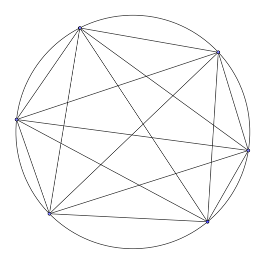
Craig: My pleasure. (*drawing and counting*)



Ooops! 31 not 32.

Jenifer: That's strange as Isaac Asimov[10] said we like to say.

[10] Perhaps apocryphal, but the reference is to a quote often attributed to Asimov to the effect that "Most of the great discoveries don't begin with 'Eureka!' but with 'Wow, that's strange. . . ' "



Craig: Usually strange is a sign there's something interesting to do. If it turned out for be powers of 2, that'd be too bad. But lets check we counted right anyway. (*recounts*) Nope, that's right. 31.

Jenifer: So I'm looking at a picture like we were drawing with the triangle game. And we want to know how many regions there.

Craig: So all we need to do is count the dots and the curves.

Jenifer: Dots is easy since we start with n of 'em on the circle.

And then there's the new ones created when two chords intersect.

Craig: Well $\binom{n}{2}$ chords so

$$\binom{\binom{n}{2}}{2}$$

pairs of chords. . .

Jenifer: Nice! But Ah! — not all pairs of chords intersect, so that won't work.

Craig: Darn! Silly of me.

Take any 4 dots on the circle. There's 4 chords that don't intersect and. . .

Jenifer: . . . 2 that do. Any 4 dots make a quadrilateral and the diagonals of the quadrilateral intersect creating a new dot.

Craig: And any new dot — we follow the chords to the circle and find the quadrilateral.

Jenifer: So there's exactly $\binom{n}{4}$ new dots for a total of $n + \binom{n}{4}$ dots in the picture.

Craig: Now edges. Ummm.

I'm thinking we count the edges going in to each dot.

Jenifer: Oh yeah — and divide by 2. Two ends to an edge, three sides to a triangle. Like that.

Much better idea than mine because I was trying to divide the edges up into different types. So for the original n dots on the circle they're connected by chords to the other $n - 1$ dots, plus they have two arcs of the circle ending at them.

Craig: For a total of $n + 1$ curves ending at each. And the new dots, inside, have 4 edges running into them since they divide their chords into two pieces each.

For a total, all together, of

$$n(n+1) + 4\binom{n}{4}.$$

Jenifer: Giving us half that many curves in the picture.

So $R + D - C = 1$ tells us that, ummm...

Craig: It says

$$R = 1 + C - D = 1 + \frac{1}{2}\left(n(n+1) + 4\binom{n}{4}\right) - \left(n + \binom{n}{4}\right).$$

Jenifer: There. I'm glad we figured that out.

Craig: But you really are sort of curious, aren't you? About what that is. I know you are. You're just hoping I'll do it.

Jenifer: I knew I could count on you.

Craig: (*mumbles as he writes*)

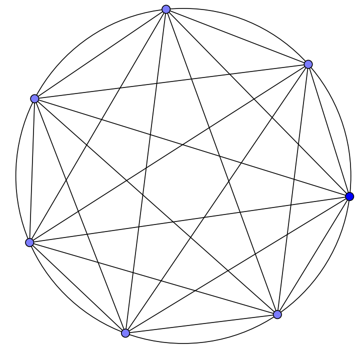
$$1 + \frac{1}{2}\left(n(n+1) + 4\binom{n}{4}\right) - \left(n + \binom{n}{4}\right) =$$
$$1 + \binom{n+1}{2} - n + \binom{n}{4} = 1 + \binom{n}{2} + \binom{n}{4}$$

$$\binom{n+1}{2} - n = \frac{(n+1)n}{2} - \frac{2n}{2} = \frac{n(n-1)}{2} = \binom{n}{2}$$

Jenifer: So?

Craig: Beg for it.

Jenifer: If n is 6? Please and thank you.



Craig: Well, $\binom{6}{2} = \binom{6}{4}$. Right? Pick 4 to include or pick 2 to leave out.
And $30/2 = 15$.

So $1 + 2(15) = 31$. Of course.

Discussion

The counting problem Jenifer and Craig discuss here can be solved in a number of ways. The approach they take uses a fact they discovered in R, C, D on page 319.

How many regions are there in this picture?