Rational Tangles

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1 Introduction

This is originally from John Conway but came to me through Tom Davis (in fact, must of this writeup is "stolen" from Tom's work: [1]). The idea is to associate a rational number to a tangle of two ropes by performing a sequence of two simple operations. Similarly, we can untangle these ropes using these same two simple operations. Hopefully along the way the students will even get some practice with fractions.

2 Getting started

Materials:

1. Nice heavy ropes about 10 feet long are about right (I use climbing straps purchased at a store like REI).

2. 2-3 plastic grocery bags.

Have four students (A, B, C, D) and have them hold two ropes as in Figure 1.

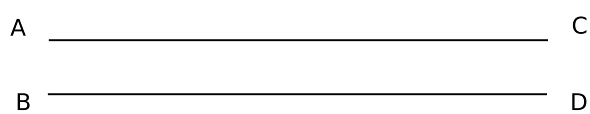

Figure 1: Initial State

Everyone needs to hold the ropes firmly. Students like the shake the ropes and everything will have to be redone if a student drops a rope. Do not allow the kids jerk on the ropes. Generally, try to keep a handle on the silliness that will result from the ropes. It is a good idea to swap kids out periodically from time to time as well.

3 The basic operations

There are two basic operations: *Twist* and *Rotate*. To twist, student D walks under the rope that student C is holding. This is the only twisting move that is allowed. There is no "untwist" move (that would undo the twist). See Figure 2 to see the result of 0, 1, 2, and 3 twists.

To rotate, students all rotate one position clockwise, as in Figure 3.

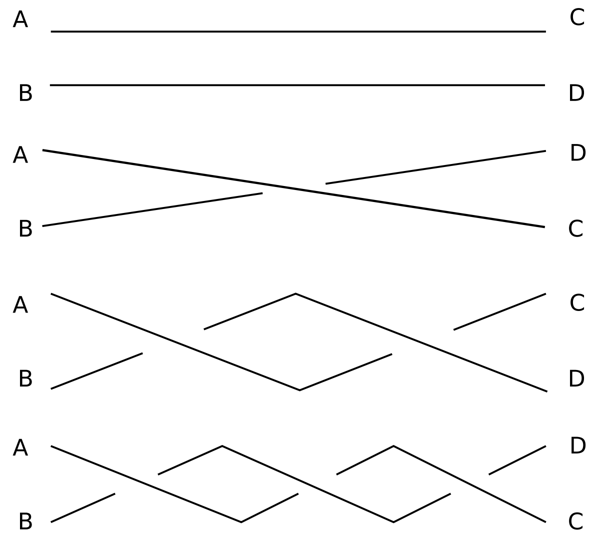

Figure 2: Twisting

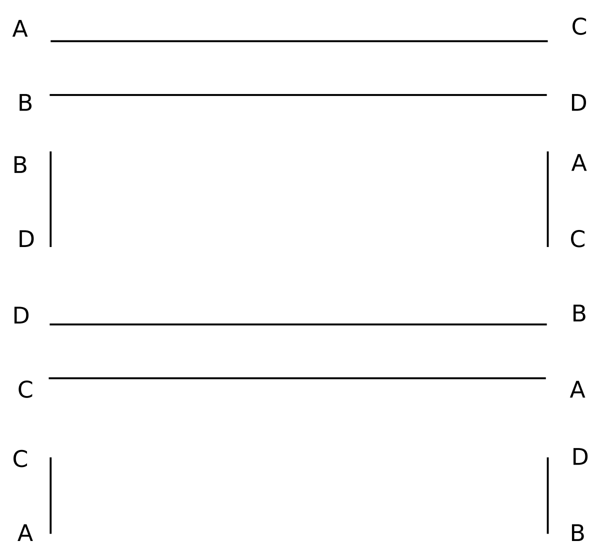

Figure 3: Rotating

We do not actually care about what position the students are in. What we care about is the position of the ropes. So, in Figure 3 the first and third positions are the same (even though the ropes have actually changed places). Similarly, the second and fourth positions are considered the same.

In describing a sequence of moves, we will write "T" for *twist* and "R" for *rotate*.

Finally, there is the *display* operation where the 4 students hold the twisted rope up for all to see.

We will write a sequence of moves by writing something like $TTRT$ to mean *twist*, *twist*, *rotate* and then *twist*, in that order.

4 Activities

One goal is to associate a number to each tangle. Here are a couple "rules" to get us started to determine how to do this.

- The starting position is given the number 0.
- Each time a *twist* is done, the number increases by 1 (so the number is an attempt to measure the number of twists made).

1. **What mathematical operation is R?**

   Start at 1 (by doing $0 \xrightarrow{T} 1$). Then perform two rotates and end back at 1:

   $$1 \xrightarrow{RR} 1$$

   What mathematical operations can do this? Try the following and determine what numbers belong at the question marks:

   $$0 \xrightarrow{T} 1 \xrightarrow{R} ? \xrightarrow{T} ?$$
   $$0 \xrightarrow{TT} 2 \xrightarrow{R} ? \xrightarrow{TT} ?$$

   Determine what mathematical operation is represented by a *rotate*.

2. **How do you get back to zero?**

   Here our goal is to start with a tangle and get it back to the 0-tangle. So, start with a tangle and try to untangle it. See if you can find a way to do this. Along the way you should learn that doing two *rotates* in a row is not productive.

   A good starting point is the tangle $TTRTTTRT$ represented by $\frac{3}{5}$. This is good because the numerator and denominator are relatively small but still complicated enough. Note:

   - Doing a *twist* to $\frac{3}{5}$ only moves the tangle further away from 0, so perhaps a *rotate* is better: $\frac{3}{5} \xrightarrow{R} -\frac{5}{3}$
   - At this point the only reasonable move is a twist since another rotate will undo the previous rotate: $-\frac{5}{3} \xrightarrow{T} -\frac{2}{3}$

- Now, a RT moves you further from 0, so it makes sense to *twist*: $-\frac{2}{3} \xrightarrow{T} \frac{1}{3}$
- Now, we just keep doing this

$$\frac{3}{5} \xrightarrow{R} -\frac{5}{3} \xrightarrow{TT} \frac{1}{3} \xrightarrow{R} -3 \xrightarrow{TTT} 0$$

So, what is the procedure and is it guaranteed to work?

3. **Infinity**

Try this:

$$0 \xrightarrow{R} ? \xrightarrow{R} 0$$

What number must belong at the question mark?

4. **GCD: Greatest Common Divisor**

There is the Euclidean algorithm for computing GCD. Here it is for the GCD of 4004 and 700

$$\begin{aligned} 4004 =& 700 \times 5 + 504 \\ 700 =& 504 \times 1 + 196 \\ 504 =& 196 \times 2 + 112 \\ 196 =& 112 \times 1 + 84 \\ 112 =& 84 \times 1 + 28 \\ 84 =& 28 \times 3 \end{aligned}$$

which shows $GCD(4004, 700) = 3$.

Note that the Euclidean algorithm still works if we use negative numbers in our calculations. Here it is for $GCD(5, 17)$:

$$\begin{aligned} 5 =& 17 \times 1 - 12 \\ 17 =& 12 \times 1 + 5 = 12 \times 2 - 7 \\ 12 =& 7 \times 1 + 5 = 7 \times 2 - 2 \\ 7 =& 2 \times 1 + 5 = 2 \times 2 + 3 = 2 \times 4 - 1 \\ 2 =& 1 \times 1 + 1 = 1 \times 2 + 0 \end{aligned}$$

Watch this:

$$-\frac{5}{17} \xrightarrow{T} \frac{12}{17} \xrightarrow{R} -\frac{17}{12} \xrightarrow{TT} \frac{7}{12} \xrightarrow{R} -\frac{12}{7} \xrightarrow{TT} \frac{2}{7} \xrightarrow{R} -\frac{7}{2} \xrightarrow{TTTT} \frac{1}{2} \xrightarrow{R} -2 \xrightarrow{TT} 0$$

What is going on? Why are these two operations so similar?

5. **What tangle numbers are possible?**

Lets start easy. Can you start with 0 and get to -3?

References

[1] Davis, Tom *Conway's Rational Tangles*, `http://www.geometer.org/mathcircles/`, 2007