Big products

Sally: You weren't there today so Jenifer just told me a puzzle and we left.

Susan: Sorry. Dentist. I forgot to tell you. What's the question?

Sally: Word for word?

> Using the digits 1,3,5,7,9 each exactly once write down numbers A and B so that the product $A \times B$ is as large as possible.

Susan: And as small as possible, right?

Sally: Sure. Why not?

Susan: Hang on, I think I know how to do that.

(*Types on laptop and then shows the results*)

How is this:

Here the python function `permutations` generates all possible rearrangements of the list $1, 3, 5, 7, 9$.

code 1 Susan's Python program to arrange digits in two numbers so their product is as big as possible.

```
from itertools import permutations
ourMin=99999
ourMax=0
a=[1,3,5,7,9]
for p in permutations(a):
    x=100*p[0]+10*p[1]+p[2]
    y=10*p[3]+p[4]
    product=x*y
    if product > ourMax:
        ourMax=product
        argmax=(x,y)
    if product< ourMin:
        ourMin=product
        argmin=(x,y)
print "Max",argmax,ourMax
print "Min",argmin,ourMin

Max (751, 93) 69843
Min (379, 15) 5685
```

Sally: I agree it seems obvious that we want a 3 digit and a 2 digit number, that's what your program assumes isn't it? And also obvious that the higher place values in each number should have the larger digits.

Susan: Well,

$$abcd \times e = ae \times 10^3 + be \times 10^2 + ce \times 10^1 + de$$
$$abc \times ed = ae \times 10^3 + be \times 10^2 + ce \times 10^1 + d \times (abc)$$

And $d \times abc > de$.

Sally: So this argument says that if we have digits to distribute among 2 numbers and the product is a maximum then the numbers have the same length or the lengths differ by just 1.

Because otherwise we could move the lowest digit of the longer number to the shorter and increase the product.

Susan: That could well be right. I just thought it said that the maximum product isn't from multiplying a 4 and a 1 digit number. Hence 3 and 2.

Sally: And the digits in a single number have to decrease.

Susan: Sure. Otherwise we could make one factor bigger without touching the other factor.

Sally: Lets try a simpler situation first. Three distinct digits a, b, c.

Susan: None of them is 0. We're guessing that the $1, 3, 5, 7, 9$ in the original question isn't important?

Sally: For the time I guess that's what we guess.

$$ab \times c = a \cdot c10 + bc$$

Susan: Swap a, b and the improvement — increase in the product — change — is

$$10bc + ac - (10ac + bc) = 10c(b - a) - c(b - a) = 9c(b - a)$$

Sally: So we get an improvement only when $b > a$.

Susan: So we can conclude that $a > b$ when the product is maximized because otherwise we could swap a and b and get something $9c$ times bigger.

Sally: We knew this already didn't we? I think this shows again that, when we have the numbers giving the maximum product, the digits in each number decrease from left to right.

Susan: Yes, but I want to be systematic in hopes of spotting a pattern that'll let us handle more complicated situations.

Sally: Like more digits. Of course.

Susan: Swap a and c and the improvement is

$$10ac + ab - (10ac + bc) = b(a - c)$$

Sally: We get better only when $a > c$ so we can conclude that $a < c$ when the product is a maximum.

And if we swap b and c the improvement is

$$10ab + bc - (10ac + bc) = 10(b - c)$$

and things get better unless $c > b$.

Susan: So we have to conclude that we've done the best we can when

$$\begin{aligned} a &> b \\ c &> a \\ c &> b \end{aligned}$$

Sally: That says $c > a > b$. So the maximum using the digits $1, 3, 5$ is 31×5.

Umm $31 \times 5 = 155$. And $51 \times 3 = 153$. Seems right. I always like to check things. I also notice that something that I was thinking would be true isn't: that the largest digit goes in the highest place was wrong.

Susan: Oh, yes. Of course the computer's example shows that as well! Things are usually much more interesting than they seem, aren't they?

Sally: As a shortcut to avoid actually calculating these inequalities I propose we can think about it this way. Here I'm comparing the right hand digits.

$$3|1 \times 0|5 \quad \text{versus} \quad 3|5 \times 0|1$$

To know which product is bigger we only need to know the sizes of the left hand most digit where the products differ. In this case we can chose between 15 tens on the left and 3 tens on the right.

Susan: I'm guessing that with longer numbers we will be getting more complicated inequalities with more than two of the variables in at least some of them. So indulge me for a little while?

Sally: Carry on!

Susan: Well how about $ab \times cd$?

Now we're going to get 6 inequalities because we can swap 6 pairs of digits. ab, ac, ad, bc, bd, cd.

Sally: But swaps in a given number are easy and we already know about them.

Susan: Well

$$ba \times cd - ab \times cd = cd(10(b-a) + (a-b)) = 9cd(b-a)$$
$$ab \times dc - ab \times cd = 9ab(d-c)$$

So to ensure that these aren't improvements we need

$$a > b$$
$$c > d$$

Sally: Okay. Kinda obvious I guess. Four more to go.

Swapping a and c is the same as swapping b and d isn't it? So the inequality we get will have to have all 4 of them in it. Appearing symmetrically somehow.

Susan: Nice observation. Or is it a guess? Let's find out. Swapping a and c we calculate

$$cb \times ad - ab \times cd$$

If we can get this one to work I bet we can get them all.

Sally: Well,

$$\begin{aligned} 100ac + 10(ab+cd) + bd - 100ac - 10(bc+ad) - bd &= 10(ab+cd-bc-ad) \\ &= 10b(a-c) + 10d(c-a) \\ &= 10(b-d)(a-c) \end{aligned}$$

To have the maximum product at $ab \times cd$ we need that $(b-d)(a-c) < 0$.

Yuck!

Susan: Oh.

(*Thinks a bit*)

But in this case, symmetry, we can just assume that $a > c$. Otherwise we just interchange the numbers. So then $b < d$ and we have:

$$a > b$$
$$c > d$$
$$d > b$$
$$a > c$$

Sally: So $a > c > d > b$. That's it?
Using the digits $1, 3, 5, 7$ that says the biggest product is 71×53.

Susan: And now as you were saying, we can sort of see this easily. Once we know it looks like $7x \times 5y$ we just chose between having 3 7's and 1 5 and the alternative which is 3 5's and 1 7.

Sally: How do we know it doesn't look like $75 \times xy$?

Susan: Then it'd have to be 75×31. Five threes and 1 seven or five sevens and three fives?

Sally: Just putting in a word for my approach, suppose $ab \times cd$ is the maximum product. Think about it this way:

$$a|b \times c|d \quad \text{versus} \quad a|d \times c|b$$

Because $a > c$ we know $d > b$. The other inequalities we derived are easy since digits decrease in numbers and we can assume $a > c$.

It could be that there's some more rules we could apply to gradually increase a product by swapping the locations of digits. But I'd rather work out all the inequalities for another round of swaps to see if we can do that in general.

Susan: That makes sense to me. We'd probably have to do something like that to justify a batch of maximizing swap rules anyway. So $abc \times de$? That was Jenifer's original problem.

Sally: My approach starts with the obvious $a > b > c$, $d > e$ and $a > d$. And then it says things like that the choice

$$a|bc \times 0|de$$

tells us $de > bc$ and so $d > b$.

Susan: And

$$ab|c \times 0d|e$$

tells us $e > c$.

Sally: You got it! So we know

$$a > d > b > c$$

and the only question is the relative positions of b and e.

Susan: So we make this comparison

$$abc \times de \approx 100d(ab)$$
$$aec \times db \approx 100d(ae)$$

Since the first is supposed to be the maximum product, we see that

$$b > e.$$

Sally: And so

$$a > d > b > e > c.$$

Using the digits $1, 2, 3, 4, 5$ we say the biggest product is 531×42.

That's what your program says isn't it? With different digits?

Susan: Sorry, wait.

No. It says 431×52.

Sally: Mind over machine, Susie!

$531 \times 42 = 22302$

Susan: And $431 \times 52 = 22412$.

So there. Machine over mind, sometimes, Sally!

Sally: Man! "When life looks like Easy Street there is danger at your door."

Susan: What?

Sally: Something I heard Mr. Phelps say once.

Susan: So what've we done wrong? We must have made a mistake.Let's go over it more carefully.

Sally: Nice you said "we!"

To start $a > b > c$ and $d > e$.

Susan: Agree.

Sally: And then obviously $a > d$. Higher place value.

Susan: Exactly why is $a > d$?

Sally: Oh my! That was an assumption we later rejected wasn't it? Sorry about that.

Let me see. To find out if $a > d$ we need to compare

$$abc \times de \quad \text{and} \quad dbc \times ae.$$

And ask when $db + ea > ab + ed$.

That says $(d - a)(b - e) > 0$.

Susan: And asking whether $b > e$ we have to compare

$$abc \times de \quad \text{with} \quad aec \times db.$$

Sally: And again we get $db + ae > de + ba$.

Susan: So either $d > a$ and $b > e$ or $d < a$ and $b < e$. We have to check those to cases to see which works.

Sally: But we know now that $d > a$ gives us the maximum product. It'd be nice to see in one step why the biggest digit is in the shorter number. At least that seems to be the pattern.

Susan: Okay. Some care is required and we can't do just short cuts, I don't think. But I have to agree that your quick approach is easier for numbers of the same length. And the key observation — that we just need to compare most significant digits that differ — that's really nice and saves pointless algebra with lower powers of 10. It seems that we've said how to solve this sort of problem but not what the solutions are. That isn't a very general understanding and sort of leaves me uncomfortable.

Sally: Then again, there's only 9 digits so only a small number of these puzzles. What's to be general about?

Susan: One more? Just for fun? $abc \times def$?

Sally: Why not? $a > b > c$ and $d > e > f$ and $a > d$ to start with. Because abc and def can be interchanged we can assume $a > d$.

Susan: Then $e > b$ and $f > c$. So we have

$$a > d > e > b > f > c.$$

Sally: So

$$631 \times 542.$$

Want to check that with your program?

Susan: Not really.

I guess I'm done with this for a while.

Sally: Me too, but there is the question of other number bases.

Susan: Same trick works.

And minimizing the product.

Sally: And products of more than two numbers.

Susan: Still, the thrill is gone.

Sally: I have to agree. What now?

Susan: What else? Chocolate!

Discussion

1. What do you do to minimize the product?

2. Of all the rectangles with a given perimeter the one with the largest area is the square. This suggests that when there's constraints we can hope to maximize a product by selecting the factors to be as nearly equal as possible. Does this make any sense in the context of Jenifer's problem?

3. Is Sally's guess "... if we have digits to distribute among 2 numbers and the product is a maximum then the numbers have the same length or the lengths differ by just 1" true?

4. Is Sal and Susie's guess — that the exact digits used in making the numbers don't matter as long as they are non-zero and distinct and the best arrangement only depends on the relative sizes of the digits— correct?

5. Check out the `permutations` command in Python by looking at the output of a little script like:

```
from itertools import permutations
a=[1,2,3]
for p in permutations(a):
    print p
```