Instructions: Work in small groups on the following problems. Try to stay together and help each other. Don't ruin the discovery for others. There won't be time for you to finish all of the problems, but I hope you will be intrigued enough to continue investigating.

1. Do "perfect shuffles" restore the order of decks with other sizes? If not, why not? If so, what stays the same and what changes? Record what happens on the chart paper. (**Note:** Remember that a "perfect shuffle" interleaves cards perfectly while keeping the top card on top.)

2. Start with an 8-card deck arranged in some sensible order. Keep track of the card second from the top as you perform "perfect shuffles." Where does this card travel? What about for a 10-card deck?

3. Arithmetic in *mod 10* is fun and easy because you only use the numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. After any calculation you only report the remainder after dividing by 10. For example, in mod 10, $6 + 5 = 1$ because 1 is the remainder when $6 + 5$ is divided by 10.

   Answer all these questions in mod 10.

   (a) $2 + 2 = \square$

   (b) $3 \cdot 4 = \square$

   (c) $\square + 5 = 2$

   (d) $4 \cdot \square = 2$

   (e) $5 \cdot \square = 3$

   (f) $\square^4 = 1$

4. Complete these addition and multiplication tables in mod 7.

+	0	1	2	3	4	5	6
0							
1							
2				5			
3							
4							
5			0				
6							5

×	0	1	2	3	4	5	6
0							
1							
2				6			
3	0						
4							
5			3				
6							1

5. Repeat Problem 3, except this time do the arithmetic in mod 7 instead of mod 10.

6. (a) Find the repeating decimal for $\frac{1}{41}$. List all the remainders you encountered during the long division, starting with 1 and 10.

   (b) Write $\frac{100}{41}$ as a mixed number.

   (c) Find the repeating decimal for $\frac{18}{41}$. List all the remainders you encountered, including 1 and 10.

   (d) Find the repeating decimal for $\frac{1}{37}$. List those remainders!

   (e) Find the repeating decimal for $\frac{1}{27}$. Cool!

7. What do these equations have to do with $\frac{1}{41}$?

$$\begin{aligned} 1 &= 0 \cdot 41 + 1 \\ 10 &= 0 \cdot 41 + 10 \\ 100 &= 2 \cdot 41 + 18 \\ 180 &= 4 \cdot 41 + 16 \\ 160 &= 3 \cdot 41 + \\ &= 9 \cdot 41 + 1 \end{aligned}$$

Finish the equations, then use the same method to find the decimal expansion of $\frac{2}{13}$.

8. Complete this table. Splitting up the work is a great idea, but don't use fancy spreadsheets or computer programs. Instead, think carefully about what mathematical shortcuts you could use to work less.

n	Powers of 10 in mod n	Cycle Length
41		
37		
3		
7		
11		
13	$1, 10, 9, 12, 3, 4, 1, \ldots$	6
17	$1, 10, 15, 14, 4, 6, 9, 5, 16, 7, 2, 3, 13, 11, 8, 12, 1, \ldots$	
21		
51	$1, 10, 49, 31, 4, 40, 43, 22, 16, 7, 19, 37, 13, \quad, \quad, \quad, \quad, \ldots$	

9. What do the the powers of 10 in mod n have to do with the decimal expansion of $\frac{1}{n}$?

10. What do these equations have to do with the base 2 "decimal" for $\frac{1}{21}$?

$$\begin{aligned} 1 &= 0 \cdot 21 + 1 \\ 2 &= 0 \cdot 21 + 2 \\ 4 &= 0 \cdot 21 + \\ 8 &= \quad \cdot 21 + 8 \\ 16 &= 0 \cdot 21 + \\ 32 &= 1 \cdot 21 + \\ 22 &= 1 \cdot 21 + \end{aligned}$$

Finish the equations above.

11. Complete this table. Again, splitting up the work is a great idea, but don't use fancy spreadsheets or computer programs. Instead, think carefully about what mathematical shortcuts you could use to work less.

n	Powers of 2 in mod n	Cycle Length
7	$1, 2, 4, 1, 2, 4, 1, \ldots$	3
9		
11		
13		
15		
17		
19		
21		
23		
25		
27		

12. Without doing any long division, predict the length of the base-2 "decimal" expansion of $\frac{1}{27}$. Confirm your answer.

13. (a) Here is a deck of 12 cards. Write out the order of the cards after a few "perfect shuffles."

1	2	3	4	5	6	7	8	9	10	11	12

(b) Here is a very similar looking deck of 12 cards! Write out the order of the cards after a few "perfect shuffles."

0	1	2	3	4	5	6	7	8	9	10	11

14. What are the powers of 2 in mod 51?

15. Explain why a 52-card deck will return to its original position after 8 perfect shuffles.

16. Complete this table. A number x is a *unit* in mod n if there is a number y such that $xy = 1$. Make sure to compare your work with others on this one because it's really easy to skip a number.

n	Units in mod n	# units in mod n
8	$1, 3, 5, 7$	4
15	$1, 2, 4, 7, 8, 11, 13, 14$	8
24		
25		
30		
49	*too many to list*	
67	*too many to list*	

17. What is an easy way to determine if a number x is a unit in mod n?

18. Complete this table. You can use long division in base 2, or the method suggested by problem 10.

n	"Decimal" for $1/n$ in base 2	# of repeating digits	# of units in mod n
3	$0.\overline{01}$	2	2
5			
7			
9			
11			
13			
15			
17			
19			
21	$0.\overline{000011}$	6	

19. Oh my goodness. This is a huge, obnoxious table. *Think about how to fill out this table using as little work as possible!*

Fraction	Base 2 "Decimal"	Fraction	Base 2 "Decimal"
1/51	$0.\overline{00000101}$	26/51	
2/51		27/51	
3/51		28/51	
4/51		29/51	
5/51		30/51	
6/51		31/51	
7/51		32/51	
8/51		33/51	
9/51		34/51	
10/51		35/51	
11/51		36/51	
12/51		37/51	
13/51		38/51	
14/51		39/51	
15/51		40/51	
16/51		41/51	
17/51		42/51	
18/51		43/51	
19/51		44/51	
20/51		45/51	
21/51		46/51	
22/51		47/51	
23/51		48/51	
24/51		49/51	
25/51		50/51	$0.\overline{11111010}$

20. Here is a 52-card deck being "perfectly shuffled."

    `http://tinyurl.com/pcmi52cards`

    Follow some cards. Follow some remainders. Figure out how you can use the cards to find the *entire base-2 expansion* for a fraction in the form $\frac{n}{51}$.

21. (a) Take all the powers of 2 in mod 51 and multiply them by k. That was fun. Whee!
    (b) A 52-card deck returns to its original position in 8 "perfect shuffles." But some cards return sooner. Find an equation that would be true for any card k that returns to its original position after 2 shuffles, then solve it.
    (c) Find an equation that would be true for any card k that returns to its original position after 3 shuffles, then solve it.
    (d) FOUR!

22. *With an even number of cards*, there is a different way to do another shuffle by starting from the bottom half instead of the top. For example,

    $$123456 \Rightarrow 415263$$

    This is called an "in-shuffle." (What we have been calling a "perfect shuffle" is sometimes also called an out-shuffle.) What changes? Determine the number of in-shuffles needed to restore the order of decks of different sizes.

23. (a) Here is a deck of 10 cards. Write out the order of the cards after a few in-shuffles.

1	2	3	4	5	6	7	8	9	10

    (b) Why might you want to number the cards starting from 0 or 1 for a particular type of shuffle? Does anything change with an odd number of cards? (Compare this to problem 13.)

24. (a) Perform in-shuffles on a 16-card deck, tracking the position of the first card.
    (b) Do it again for a 12-card deck. Where have you seen these numbers before?
    (c) Without using any cards, predict what positions the top card of a 24-card deck will take when it is in-shuffled. Then, perform the shuffles to see if you were right.

25. It turns out that you can use a lot fewer than 52 shuffles to move any card that you want to the top of a regular deck of 52 cards. In fact, there is a way to use out- and in-shuffles to move any card in the deck to the top of the deck. There are magicians who know how to do these shuffles perfectly and can use them to perform tricks. Figure out the sequence of out- and in-shuffles required to move any card in the deck to the top of the deck.

Link to this document: `http://bit.ly/nwmc-perfect-shuffles`